Name:________________________

Trig Identity Homework

Math 1220

1 Useful Trig Identities

Theorem 1 (Fundamental Trig Identities)**.** Given any numbers x, y, we have the following equalities

(a) $\sin^2(x) + \cos^2(y) = 1$ (The Pythagorean identity)

(b) $\sin(x + y) = \sin(x)\cos(y) + \sin(y)\cos(x)$ (The sine sum formula)

(c) $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ (The cosine sum formula)

You must **memorize these identities** for this class. They will show up many times throughout the semester, including on exams!

Theorem 2 (Derived Trig Identities)**.** Given any numbers x, y, we have the following equalities

(a) $\tan^2(x) + 1 = \sec^2(x)$

(b) $\cot^2(x) + 1 = \csc^2(x)$

(c) $\sin(x - y) = \sin(x)\cos(y) - \sin(y)\cos(x)$

(d) $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

(e) $\sin(2x) = 2\sin(x)\cos(x)$

(f) $\cos(2x) = \cos^2(x) - \sin^2(x)$

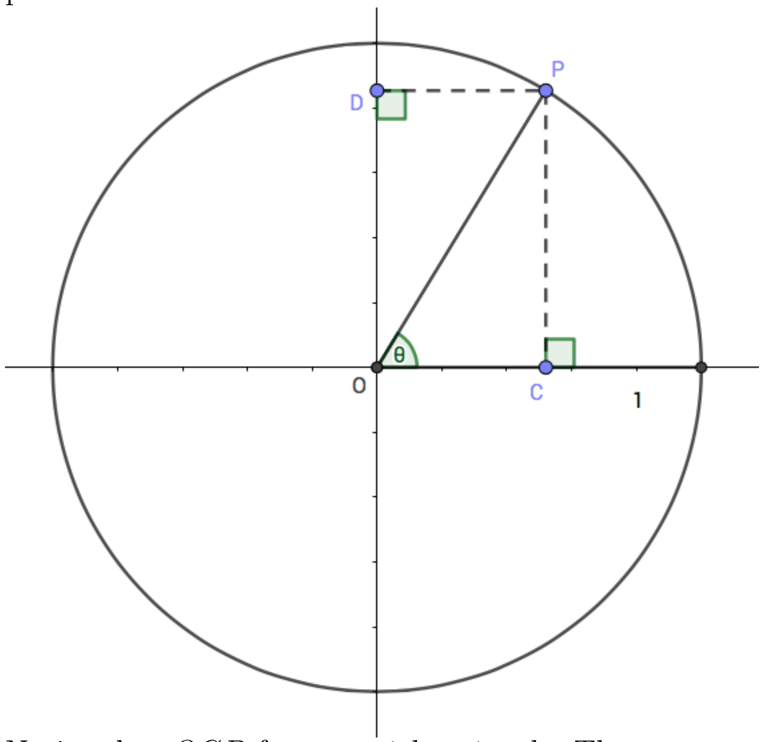
(g) $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

(h) $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

Challenge 3. To warm up with trig functions, find **all** values of θ such that $\sin^2\theta = \sin\theta$. You do not need to use any trig identities, but consider solving $x^2 = x$ first.

2 Pythagorean Identity

Challenge 4. Recall from notesheet 12.2 that, if a point P is on the unit circle and forms an angle θ with the x-axis, then, in coordinates, $P = (\cos\theta, \sin\theta)$. Thus, we have the following picture:



(a) Fill in the following values in terms of θ

- The length of $OC =$
- The length of $PC =$
- (Does not depend on θ). The length of $OP =$

Notice that OCP forms a *right triangle*. Thus, we can use The Pythaogrean Theorem on the triangle OCP.

(b) Fill in the Pythagorean theorem in terms of the right triangle OCP to get the Pythagorean identity from Theorem 1.

Challenge 5. Show that $\cot(x) + \tan(x) = \sec(x)\csc(x)$ using the definitions of tan and cot, as well as the Pythagorean identity.

(a) Start with $\cot(x) + \tan(x)$ and substitute $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$. Also do the likewise substitution for $\cot(x)$.

(b) Find a common denominator for the sum of fractions and combine them.

(c) Simplify the numerator using the Pythagorean identity.

(d) Rewrite your fraction as a product of two fractions.

(e) Rewrite your expression using the definition of $\sec(x)$ and $\csc(x)$.

Challenge 6. Verify $\cot^2(x) + 1 = \csc^2(x)$. Hint: Divide the Pythagorean identity by something.

3 Sum Identities

Challenge 7. Using the fact $\sin(x + y) = \sin(x)\cos(y) + \sin(y)\cos(x)$, verify $\sin(2x) = 2\sin(x)\cos(x)$ by setting $x = y$.

Challenge 8. Using any of the listed trig identities on the first page, compute $\sin\left(-\frac{\pi}{12}\right)$.

Challenge 9. Use the sine sum formula to prove $\sin\left(x+\frac{\pi}{2}\right) = \cos(x)$.

4 Derivatives of Trig Identities

Recall the definition of the derivative:

Definition 10. Given a function $f(x)$, we say its derivative $f'(x)$ is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{provided the limit exists.}$$

Another useful fact that we will see later is the following

Theorem 11 (Useful fact about sine). $\lim_{h\to 0} \frac{\sin(h)}{h} = 1 \implies \lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$

Challenge 12. Find the derivative of $f(x) = \sin(x)$.

(a) Do the substitution $f(x) = \sin(x)$ into the difference quotient.

(b) Use a trig identity to expand $\sin(x+h)$.

(c) Distribute the $\lim\limits_{h\to 0}$ term across the plus sign and pull any terms that depend only on x outside the limit.

(d) Using the "Useful fact about sine" above, evaluate the limits.

Challenge 13. Draw a graph of $f(x) = \sin(x)$, including at least $[-\pi, 2\pi]$ in your x-axis. Draw the tangent line through the point $\left(\frac{\pi}{2}, \sin\frac{\pi}{2}\right)$ and also the tangent line through the point $(\pi, \sin\pi)$. What are the slopes of these tangent lines?

Challenge 14. Draw a graph of $g(x) = \cos(x)$, including at least $[-\pi, 2\pi]$ in your x-axis. Draw the tangent line through the point $(0, \cos(0))$ and also the tangent line through the point $\left(\frac{\pi}{2}, \cos\frac{\pi}{2}\right)$. What are the slopes of these tangent lines?

Challenge 15. Let $f(x) = \sin x$ and $g(x) = \cos x$ as above.

(a) Can you use a symmetry argument and the chain rule to see why $f'\left(x + \frac{\pi}{2}\right) = g'(x)$?

(b) Simplify $f'\left(x + \frac{\pi}{2}\right)$ to find $g'(x)$.

Challenge 16. Using what you learned above, give at least 3 examples of a function $y(x)$ such that

$$y(x) + y''(x) = 0$$