Name:________________________

Series Worksheet

Math 1220

Definition 1. Way back in chapter 7, we defined the improper integral:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \text{ is } \begin{cases}\text{Convergent if the limit exists}\\ \text{Divergent otherwise}\end{cases}$$

Let $\{a_n\}_{n=k}^\infty$ be a sequence. Recall from 8.3 that we say

$$\sum_{n=k}^\infty a_n = \lim_{N\to\infty}\sum_{n=k}^N a_n \text{ is } \begin{cases}\text{Convergent if the limit exists}\\ \text{Divergent otherwise}\end{cases}$$

Challenge 2. For the following sequences, complete the the following: $\sum_{n=1}^1 a_n, \sum_{n=1}^2 a_n, \sum_{n=1}^5 a_n, \sum_{n=1}^N a_n$ (simplify this last expression if you can)

$$(i) a_n = \frac{1}{n} - \frac{1}{n+2} \quad (ii)\ a_n = \frac{1}{n^2+2n}, \quad (iii)\ a_n = \frac{5}{2^n}, \quad (iv)\ a_n = \frac{2}{n^2}, \quad (v)\ a_n = 2^n$$

Then, if you can, find both $\lim_{n\to\infty} a_n$ and $\sum_{n=1}^\infty a_n = \lim_{N\to\infty}\sum_{n=1}^N a_n$ for the above sequences.

Challenge 3. Consider the infinite series

$$\sum_{n=0}^\infty r^n = 1 + r + r^2 + r^3 + r^4 + \cdots$$

Now, (a) compute

$$\left(\sum_{n=0}^N r^n\right) - r\left(\sum_{n=0}^N r^n\right)$$

and then (b) take $\lim_{N\to\infty}$ of your expression. Note that we can now use this technique. For example

$$\sum_{n=1}^\infty \frac{5}{2^n} = \frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \cdots = \frac{5}{2}\left(\underbrace{1 + \frac{1}{2} + \frac{1}{4} + \cdots}_{\text{Use identity with } r=1/2}\right)$$

Challenge 4. By writing out the series and manipulating them or canceling them, evaluate the series and say whether the series is convergent/divergent.

$$(i)\ \sum_{n=0}^\infty \frac{5^{n+1}}{7^n}, (ii)\ \sum_{n=0}^\infty -5\left(\frac{2}{3}\right)^n, (iii)\ \sum_{n=0}^\infty \left(\frac{2}{3}\right)^n, (iv)\ \sum_{n=1}^\infty \left(\frac{2}{3}\right)^n, (v)\ \sum_{n=1}^\infty \left(\frac{2}{3}\right)^{2n}, (vi)\ \sum_{n=100}^\infty \left(\frac{2}{3}\right)^n$$

Definition 5. Remember that a p-series has the following rule for p some constant:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ will } \begin{cases} \text{converge if } p > 1 \\ \text{diverge if } p \leq 1 \end{cases}$$

The *Harmonic series*, $\sum_{n=1}^{\infty} \frac{1}{n}$ is a special case of this situation with $p = 1$, and is therefore divergent, thus,

$$\lim_{n\to\infty} \frac{1}{n} = 0 \text{ but } \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} = \infty/\text{DNE}$$

Challenge 6. Determine if the following series converge/diverge:

$$(i) \sum_{n=1}^{\infty} \frac{1}{n^5}, (ii) \sum_{n=2}^{\infty} \frac{1}{n^5}, (iii) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}, (iv) \sum_{n=1}^{\infty} \frac{1}{n^{1.00000001}}, (v) \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n^3}$$

Theorem 7. Remember also the integral test, which says, given sequence $\{a_n\}_{n=k}^{\infty}$ and continuous, positive, decreasing function $f(x)$ such that $f(n) = a_n$ for $n \geq k$, then

$$\sum_{n=k}^{\infty} a_n \text{ and } \int_k^{\infty} f(x)\, dx$$

both "do the same thing" in terms of convergence/divergence.

Challenge 8. Figure out if the following series converge/diverge using the integral test.

$$(i) \sum_{n=0}^{\infty} ne^{-n^2}, (ii) \sum_{n=2}^{\infty} \frac{1}{n\ln n}, (iii) \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}, (iv) \sum_{n=3}^{\infty} \frac{n^2}{e^n}$$

Challenge 9. Use the comparison test to figure out if the following series converge/diverge

$$(i) \sum_{n=1}^{\infty} \frac{4+3^n}{2^n}, (ii) \sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}, (iii) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}, (iv) \sum_{n=1}^{\infty} \frac{\ln n}{n}, (v) \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$$

Challenge 10. (Old Exam Series) Determine whether each of the following series converges/diverges. If the series converges, compute its sum.

$$(i) \sum_{n=2}^{\infty} \left(\frac{1}{(\ln n)^2} - \frac{1}{(\ln(n+1))^2} \right), (ii) \sum_{n=0}^{\infty} \frac{e-(-2)^n}{3^n}, (iii) \sum_{n=5}^{\infty} \frac{4^{n+2}}{5^n}$$

Challenge 11. (Old Exam Series) Determine whether each of the following series converges/diverges.

$$(i) \sum_{n=1}^{\infty} \frac{2}{n^e}, (ii) \sum_{n=2}^{\infty} \frac{1}{n^{1/3}-1}, (iii) \frac{6n}{n^2+1}, (iv) \sum_{n=2}^{\infty} \frac{5}{n\sqrt{n^3+5}}, (v) \sum_{n=1} \frac{n^2}{n(n+1)}$$