MATH 1210
DERIVATIVE WORKSHEET

Limit Definition of the Derivative. Let f be a function and x be a real number. Then the derivative of f at x, denoted $f'(x)$ or $\frac{d}{dx}[f(x)]$, is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{provided this limit exists.}$$

If the above limit exists, we say that f is differentiable at x, otherwise we say that f is not differentiable at x.

Geometric Interpretation of the Derivative. If f is differentiable at x, then $f'(x)$ is the slope of the line tangent to the graph of f at the point $(x, f(x))$.

Rate of Change Interpretation of the Derivative. If f is differentiable at x, then $f'(x)$ is the instantaneous rate of change of f with respect to x.

Examples.

(1) If $p(t)$ is position at time t, then $p'(t)$ is instantaneous velocity at time t.
(2) If $v(t)$ is velocity at time t, then $v'(t)$ is instantaneous acceleration at time t.
(3) If $C(x)$ is the cost of producing x goods, then $C'(x)$ is marginal cost.

Differentiation Rules.

(1) Let $f(x) = c$ be a constant function. Then, $f'(x) = 0$.
(2) Let f be differentiable at x and let c be a constant. Then
$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$
(3) (Power Rule) Let n be any real number. Then,
$$\frac{d}{dx}x^n = nx^{n-1}$$
(4) (Sum Rule) Let f and g be two functions such that f and g are both differentiable at x. Then,
$$(f+g)'(x) = f'(x) + g'(x)$$
(5) (Product Rule) Let f and g be two functions such that f and g are both differentiable at x. Then
$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$
(6) (Quotient Rule) Let f and g be two functions such that $g(x) \neq 0$ and f and g are both differentiable at x. Then,
$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$
(7) (Chain Rule) Let f and g be two functions such that f is differentiable at x and g is differentiable at $f(x)$. Then,
$$(g \circ f)'(x) = g'(f(x))f'(x)$$
(8) (Generalized Power rule) Let f be differentiable at x and let n be a real number. Then
$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$$

Theorem 0.1. *Let f be a function. If f is differentiable at x, then f is continuous at x.*

An equivalent way to say the above theorem is that if f is not continuous at x, then f is not differentiable at x. This is the contrapositive of the theorem 0.1.

Warning: The converse of theorem 0.1 is not true. In other words, f being continuous at x **DOES NOT** imply that f is differentiable at x. Here is an example illustrating that:

Example. Let $f(x) = |x|$. Then f is continuous at 0, but not differentiable at 0.

Practice Problems.

(1) In each of the following problems calculate $f'(x)$.

(a) $f(x) = (x+1)^7 + x^2 + 3215$

(b) $f(x) = (x^2 + \pi)\sqrt{3x^2 + 8}$

(c) $f(x) = (-6x^8 + (2x+1)^{4/3} + 1)^{2/7}$

(d) $f(x) = \dfrac{x^2}{\sqrt{7x^4 + x^2 + 1}}$

(e) $f(x) = \dfrac{(20 - x^7 + x)^2}{1 - x}$

(2) Suppose $f(4) = -27$ and $f'(4) = 9$. Let $h(x) = f(x^3 + x - 6)$. Calculate $h'(2)$.

(3) Find an equation of the line tangent to $f(x) = 3x^{7/3} + x$ at $x = 8$.

(4) Let $f(x) = 11\sqrt{x}$. Use the limit definition of the derivative to calculate $f'(x)$.

(5) (Challenging) Let $g(x) = x^{2/3} + 4$. Use the limit definition of the derivative to calculate $g'(x)$. (Note: this problem is difficult, and would not appear on an exam)

(6) Let $h(x) = x^3$. Use the limit definition of the derivative to calculate $h'(x)$.