Compound Interest Worksheet

Math 1210

November 7, 2018

As briefly discussed in class, various types of monetary investments and debts accruse "interest."

Definition 1. *From an economics standpoint, interest is the charge for the privilege of borrowing money, usually expressed in terms of a percentage.*

Perhaps an example best illustrates what is going on.

Example 2. *Consider a savings account that gives* 1% *interest per year. Then, if you put* $\$100$ *in the savings account, the bank will pay you* 1% *of* $\$100$ *after a year. Thus, you earnings will be as in the table below*

Years after deposit	*Total balance*
0	$\$100.00$
1	$\$101.00 = 100 + 0.01 \cdot 100$
2	$\$102.01 = 101 + 0.01 \cdot 101$
3	$\$103.03 = 102.01 + 0.01 \cdot 102.01$
$\vdots$	$\vdots$
10	$\$110.46$
$\vdots$	$\vdots$
100	$\$270.48$

In general, interest does not need to be compounded annually. Interest can be compounded over any interval of time. It could be monthly, daily, even every second. The general formula for compound interest is the following:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{n \cdot t}$$

where P is the principal (initial deposit), r is the interest rate, and n is the number of times interest is applied per year, and t is the amount of time in years passed since the inital deposit.

In our example above, we could model the problem above by taking

$$P = \$100$$
$$r = 0.01$$
$$n = 1$$

Challenge 3. Based on the example above, consider the function

$$A(t) = 100\left(1 + \frac{0.01}{1}\right)^{1 \cdot t}$$

to compute the balance of a savings account with a 1% annual compounded interest rate. Check that the following evaluations are true. (You can use a calculator.)

(a) $A(0) = 100$

(b) $A(1) = 101$

(c) $A(2) = 102.01$

(d) $A(3) = 103.0301$

(e) $A(10) = 110.462\ldots$

(f) $A(100) = 270.481\ldots$

Can you see how the formula works based on the original example?

Challenge 4. I forget to pay my credit card bill of \$400 and I am charged 20% interest on my unpaid debt compounded monthly. How much money will I owe if I forget for 3 months? How much will I owe if I forget for 1 year? (Hint: remember that 3 months is 0.25 years.)

In class, I declared that we can model "countinuously compounded interest" using the formula

$$A(t) = Pe^{rt}$$

What is continously compounded interest and where did that formula come from?

Definition 5. *Continuously compounded interest is interest that is applied every moment.*

However, how many "moments" are in a year? This does not make much sense by trying to compute, but we can model such a statement using limits. If we applied interest every moment, we would get

$$A(t) = \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{n\cdot t}$$

However, if we try to apply limit laws, we see that we get the expression 1^∞, which is an indeterminate form, so we must be more clever!

Challenge 6. **Bonus!** We seek to evaluate the limit above.

(a) Consider $A(t) = P\left(1 + \frac{r}{n}\right)^{n\cdot t}$. Apply ln to both sides of the equation.

(b) Use log rules to bring down the exponent.

(c) Apply $\lim\limits_{n\to\infty}$ to both sides.

(d) Substitute $n = \frac{r}{h}$. If $\frac{r}{h} \to \infty$, what must h go to?

(e) What does this remind you of? Remember that $\ln(1) = 0$. Subtract $\frac{\ln(1)}{h}$ from your limit. Does it remind you of anything now?

(f) Remember that $\frac{d}{dx}\ln(x) = \frac{1}{x}$. Use this fact to evaluate your limit.

(g) We now know what $\lim_{n\to\infty} \ln(A(t))$ is equal to. Now, remember that $\ln(x)$ is a continuous function, so $\lim_{n\to\infty} \ln(A(t)) = \ln(\lim_{n\to\infty} A(t))$. Apply this fact.

(h) Exponentiate both sides of the equation to solve for $A(t)$. What is $A(t)$?